# Mathematics: The Core Course for A-level, Bostock & Chandler

## Chapter 7: Trigonometric Identities

### Exercise 7a

(1) $\sec^2\theta + \tan^2\theta = 6$

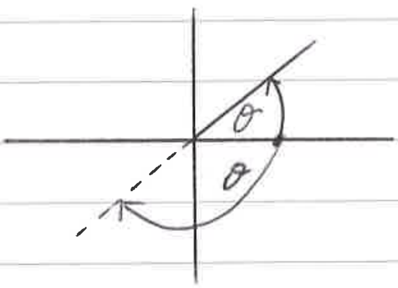
By trig id. $1 + \tan^2\theta = \sec^2\theta$ we have

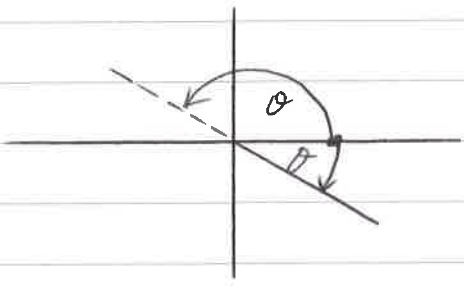
$$1 + \tan^2\theta + \tan^2\theta = 6$$

$$\therefore \tan^2\theta = \frac{5}{2} \Rightarrow \tan\theta = \pm\sqrt{\frac{5}{2}}$$

So $\theta = 57.69°, -57.69°$ (2 d.p.)

By CAST



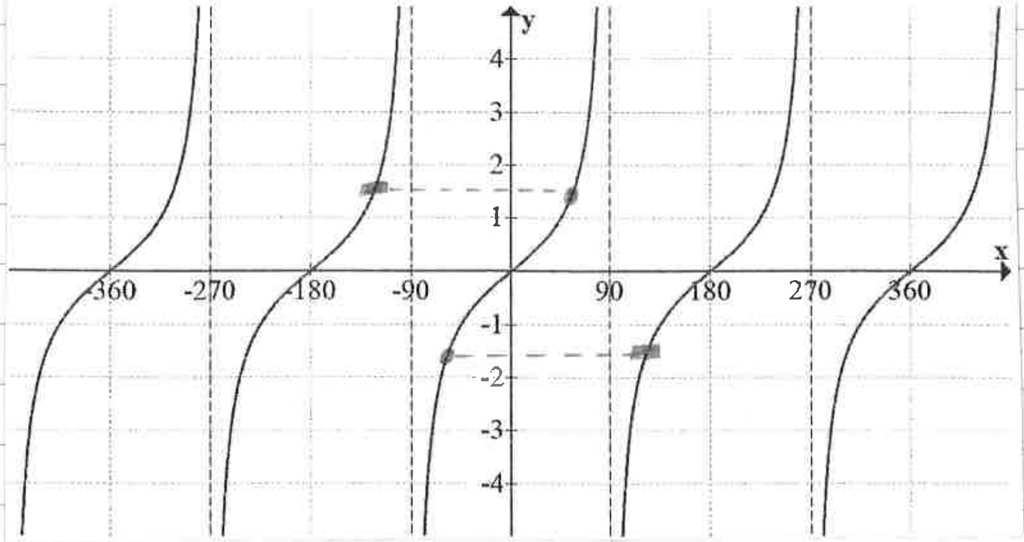




For $\theta = 57.69°$ we have also $\theta = -180 + 57.69°$
$= -122.31°$

For $\theta = -57.69°$ we also have $\theta = 180° - 57.69°$
$= 122.31°$

So $\theta = \pm 57.69°, \pm 122.31°$ in $-180° \leq \theta \leq 180°$

By graph




$\theta = \pm 57.69^\circ$ are shown as dots in the graph above

There are two other values of $\theta$ which lie in $-180^\circ \leq \theta \leq 180$

For $\theta = -57.69^\circ$ we add $180^\circ$ to get $\theta = +122.31^\circ$

For $\theta = 57.69^\circ$ we subtract $180^\circ$ to get $\theta = -122.31^\circ$

Hence $\theta = \pm 57.69^\circ, \pm 122.31^\circ$ in $-180^\circ \leq \theta \leq 180^\circ$.

(2) $4\cos^2\theta + 5\sin\theta = 3$

From $\cos^2\theta + \sin^2\theta = 1$, $\cos^2\theta = 1 - \sin^2\theta$

$\therefore \quad 4(1-\sin^2\theta) + 5\sin\theta = 3$

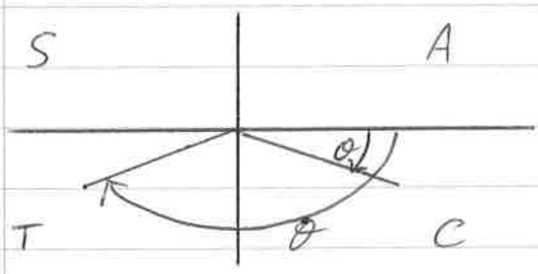
$4 - 4\sin^2\theta + 5\sin\theta = 3$

$4\sin^2\theta - 5\sin\theta - 1 = 0$

So $\sin\theta = \dfrac{5 \pm \sqrt{25+16}}{8} = 1.42539, -0.17539$

But $\sin\theta = 1.42539$ is Not valid since $-1 \le \sin\theta \le +1$

By CAST




$\sin\theta$ is $-$ve so must lie in quadrant III & IV

For quadrant IV

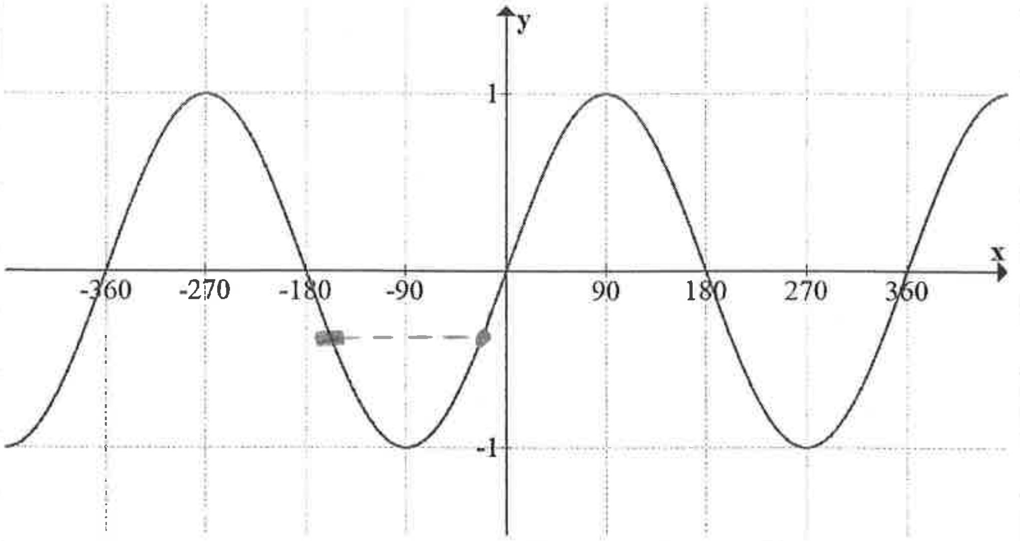
$\sin\theta = -0.17539$

$\theta = -10.1^\circ$

For quadrant III

$\theta = -180^\circ + 10.1^\circ = -169.9^\circ$

By graph




$\sin\theta = -0.17539 \Rightarrow \theta = -10.1^\circ$

$\therefore$ From the graph the only other value of $\theta$ in $-180^\circ \leq \theta \leq 180^\circ$
is $\theta = -169.9^\circ$

(3) $\cot^2\theta = \operatorname{cosec}\theta$

From $\cot^2\theta + 1 = \operatorname{cosec}^2\theta$ we have

$$\operatorname{cosec}^2\theta - 1 = \operatorname{cosec}\theta$$

$$\therefore \quad \operatorname{cosec}^2\theta - \operatorname{cosec}\theta - 1 = 0$$

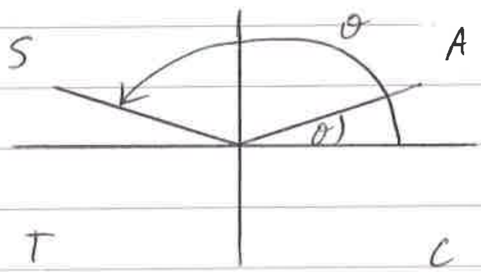
So $\operatorname{cosec}\theta = \dfrac{+1 \pm \sqrt{1+4}}{2} = \begin{matrix} 1.618033989 \\ -0.618033989 \end{matrix}$

So $\dfrac{1}{\sin\theta} = 1.618033989, -0.618033989$

So $\sin\theta = 0.61803399, -1.618033989$

But $\sin\theta = -1.618034$ is Not a valid answer.

By CAST



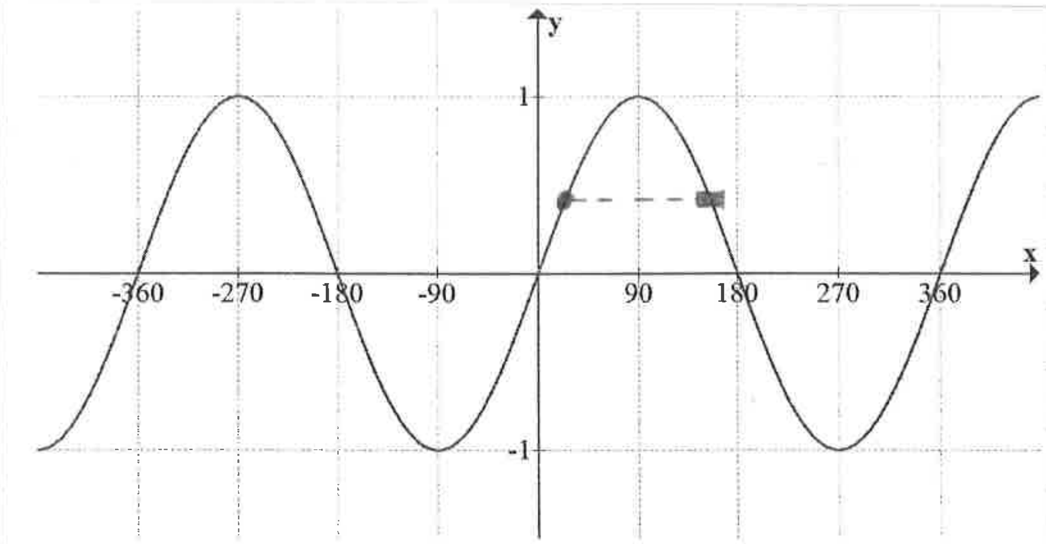

$\sin\theta = 0.61803$ is positive so must lie in quadrant I & II.

For quadrant I: $\sin\theta = 0.618033989$

$\theta = 38.173^\circ$

For quadrant II: $\theta = 180^\circ - 38.173^\circ = 141.827^\circ$

By graph




$\sin\theta = 0.618033989$ so $\theta = 38.173°$

From the graph we see that there is only one other value of $\theta$ which gives $0.618033989$, this being to the left hand side of $180°$

$\theta = 180 - 38.173 = 141.827°$

(4) $\tan\theta + \cot\theta = 2$ .

There is no trig identity for this, so convert using basic definitions of $\tan\theta$ & $\cot\theta$ :

$$\therefore \quad \frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta} = 2$$

hence $\dfrac{\sin^2\theta + \cos^2\theta}{\cos\theta \sin\theta} = 2$

$$\therefore \quad 1 = 2\sin\theta\cos\theta$$

There is a way to solve this quickly if we use something called double-angle identities, but we haven't got this far in the book. So --- ? I can't see any other way of solving this via trig.

I think the question has been put in the wrong section, & should have gone in Ex 7d. (look at 7d, No 6 which is similar)

⑤ $\tan\theta + 3\cot\theta = 5\sec\theta$

Since there are no trig identities suitable to use, convert the above using basic definitions of tan, cot, & sec:

$$\frac{\sin\theta}{\cos\theta} + \frac{3\cos\theta}{\sin\theta} = \frac{5}{\cos\theta}$$

$$\therefore \quad \sin^2\theta + 3\cos^2\theta = 5\sin\theta$$

$$\Rightarrow \quad \sin^2\theta + \cos^2\theta + 2\cos^2\theta = 5\sin\theta$$

So $\quad 1 + 2\cos^2\theta = 5\sin\theta$.

using $\sin^2\theta + \cos^2\theta = 1$ we have $\cos^2\theta = 1 - \sin^2\theta$

So $\quad 1 + 2(1 - \sin^2\theta) = 5\sin\theta$

$$3 - 2\sin^2\theta = 5\sin\theta$$

$$\therefore \quad 2\sin^2\theta + 5\sin\theta - 3 = 0$$

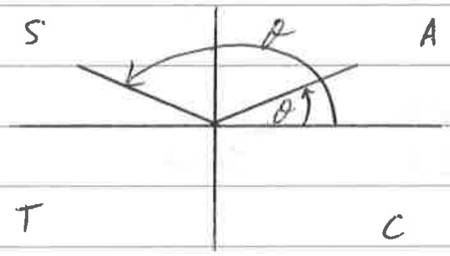
$$(2\sin\theta - 1)(\sin\theta + 3) = 0$$

Hence $\quad \sin\theta = \frac{1}{2}, -3$

But $\sin\theta = -3$ is not a valid answer since $-1 \le \sin\theta \le +1$

By CAST

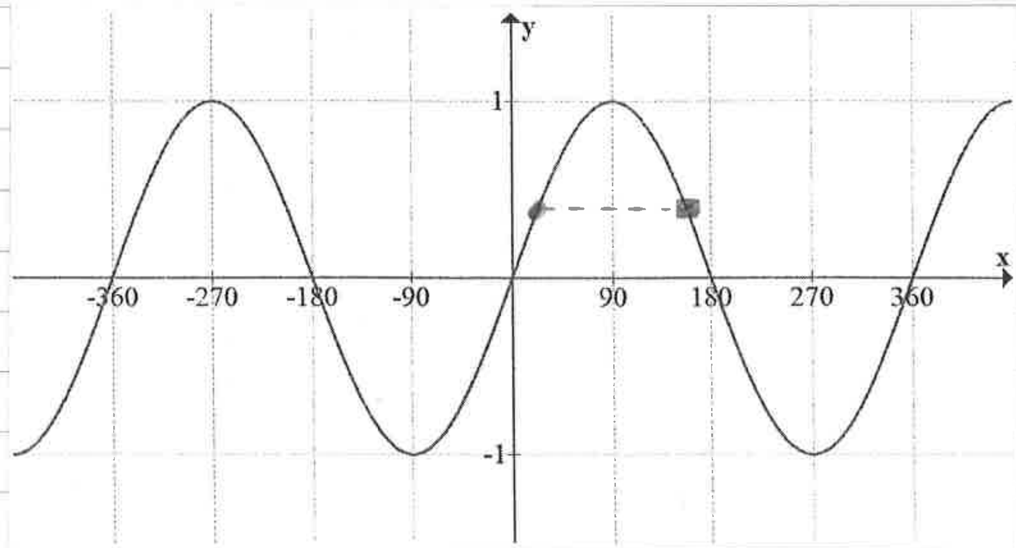
$\sin\theta = \frac{1}{2}$ is Positive so $\theta$ lies in quadrant I & II




So $\sin\theta = \frac{1}{2} \Rightarrow \theta = 30^\circ$

By CAST we have $\theta = 180^\circ - 30^\circ = 150^\circ$

By graph




$\sin\theta = \frac{1}{2} \Rightarrow \theta = 30^\circ$

From the graph we see That There is only one other Value of $\theta$ in $-180^\circ \le \theta \le 180^\circ$ which gives $\frac{1}{2}$:

$\theta = 180^\circ - 30^\circ = 150^\circ$

⑥ $\sec\theta = 1 - 2\tan^2\theta$

Using $1 + \tan^2\theta = \sec^2\theta$ we have

$$\sec\theta = 1 - 2(\sec^2\theta - 1)$$

$$= 3 - 2\sec^2\theta$$

So $2\sec^2\theta + \sec\theta - 3 = 0$

$$(2\sec\theta + 3)(\sec\theta - 1) = 0$$

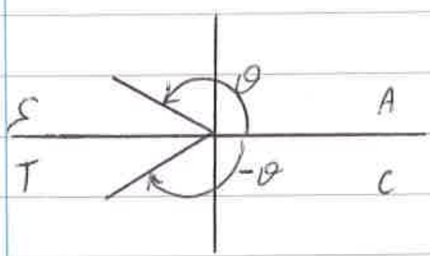
So $\sec\theta = -\frac{3}{2}, +1$

$$\frac{1}{\theta} = -\frac{3}{2}, +1$$

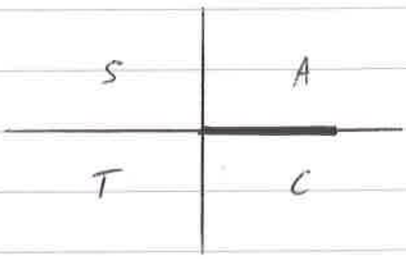
So $\cos\theta = -\frac{2}{3}, 1$

By CAST

For $\cos\theta = -\frac{2}{3}$




For $\cos\theta = 1$




$\cos\theta$ is negative so lies in quad II & III

$\cos\theta$ is positive so lies in quad I & IV

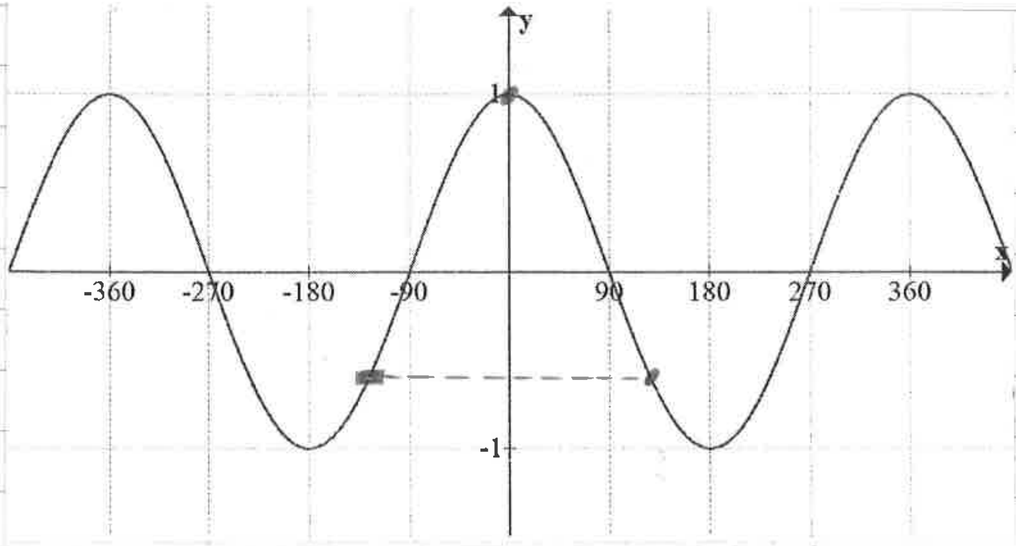
$\cos\theta = -\frac{2}{3} \Rightarrow \theta = 131.81°$

and also $\theta = -131.81°$

$\cos\theta = 1 \Rightarrow \theta = 0°$

No other values exists in $-180° \to 180°$

By gRaph




FoR $\cos\theta = -\frac{2}{3} \rightarrow \theta = 131.81^\circ$

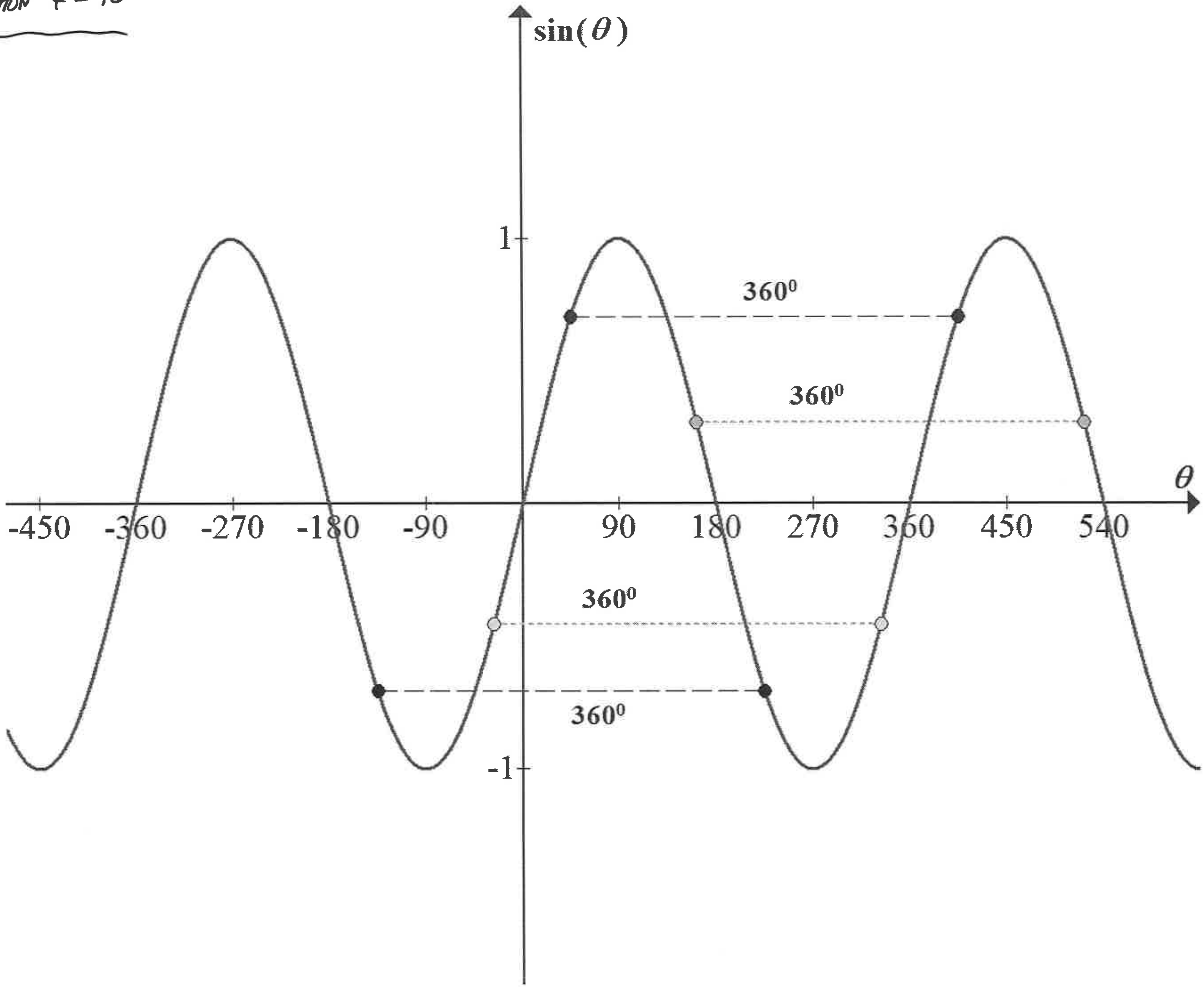
The only other value giving $-\frac{2}{3}$ lies directly opposite

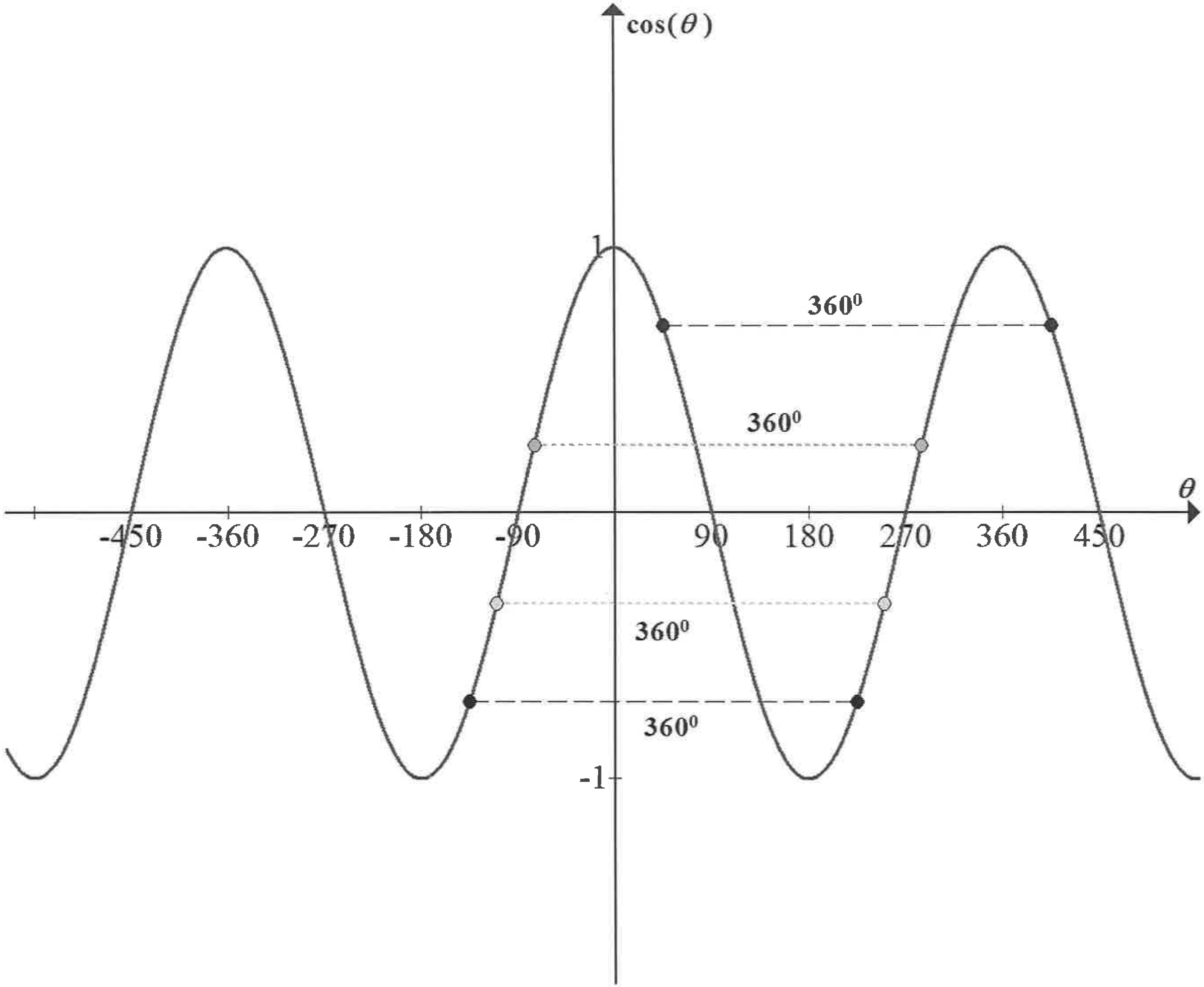
on the left hand side: $\theta = -131.81^\circ$

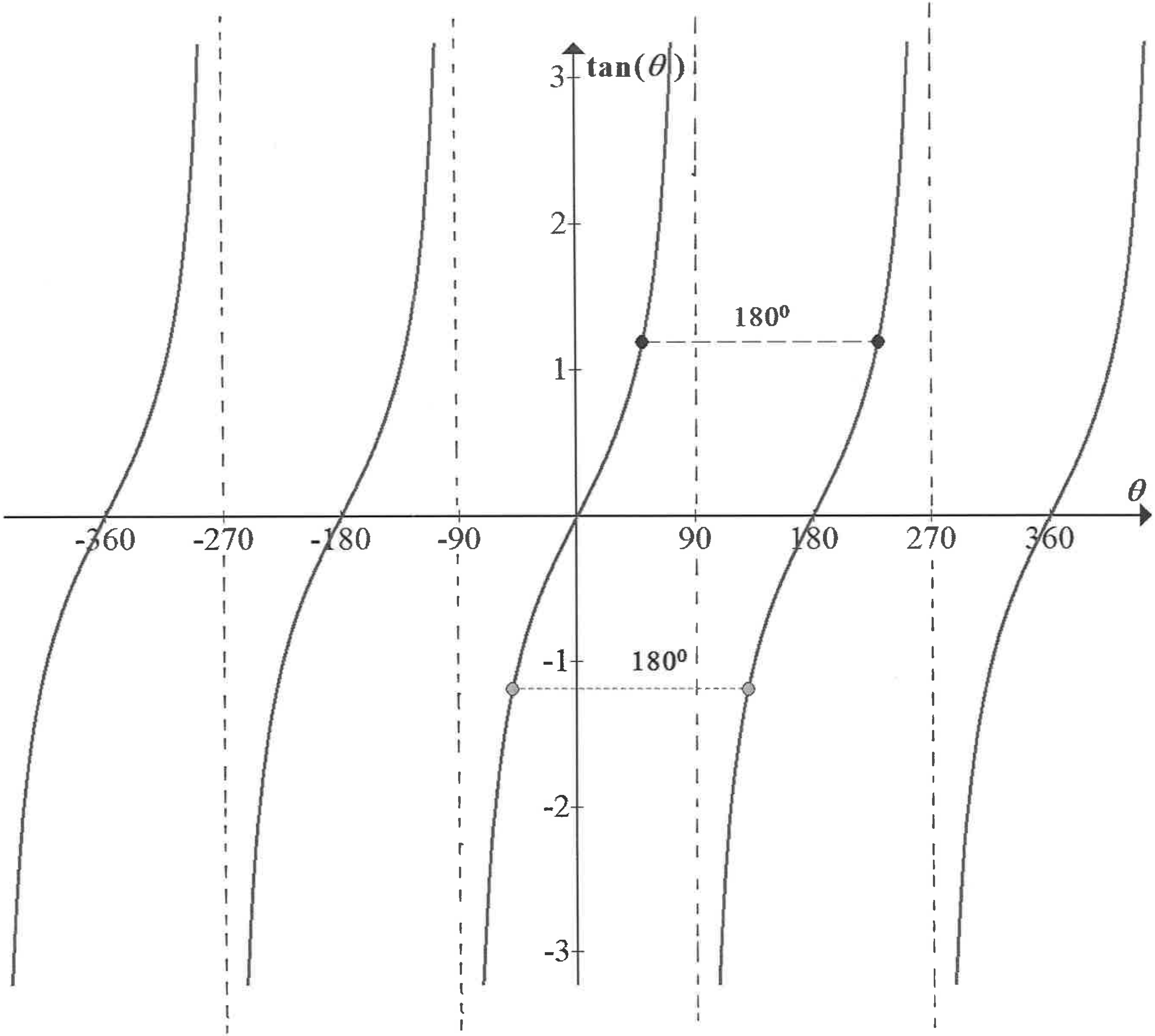
FoR $\cos\theta = 1 \rightarrow \theta = 0^\circ$

There is no other value of $\theta$ on the gRaph in $-180^\circ \leq \theta \leq 180$ which gives 1 (The next values are $360^\circ$ away)

For question 7 – 10




cos(θ)
θ
1
-1
-450
-360
-270
-180
-90
90
180
270
360
450
360⁰
360⁰
360⁰
360⁰


tan(θ)
θ
180⁰
180⁰
3
2
1
-1
-2
-3
-360
-270
-180
-90
90
180
270
360

(7) $5\cos\theta - 4\sin^2\theta = 2$

Using $\cos^2\theta + \sin^2\theta = 1$ we have $\sin^2\theta = 1 - \cos^2\theta$, so

$$5\cos\theta - 4(1-\cos^2\theta) = 2$$

$$\therefore\ 5\cos\theta - 4 + 4\cos^2\theta = 2$$

$$\therefore\ 4\cos^2\theta + 5\cos\theta - 6 = 0$$

Hence

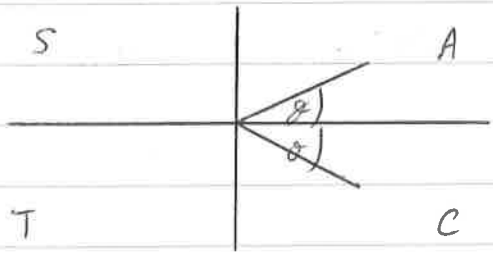
$$\cos\theta = \frac{-5 \pm \sqrt{25+96}}{8}$$

$$\cos\theta = \frac{3}{4},\ -2$$

But $\cos\theta = -2$ is not valid so $\cos\theta = \frac{3}{4}$

$$\therefore\ \theta = 41.41^\circ \pm 360^\circ n$$

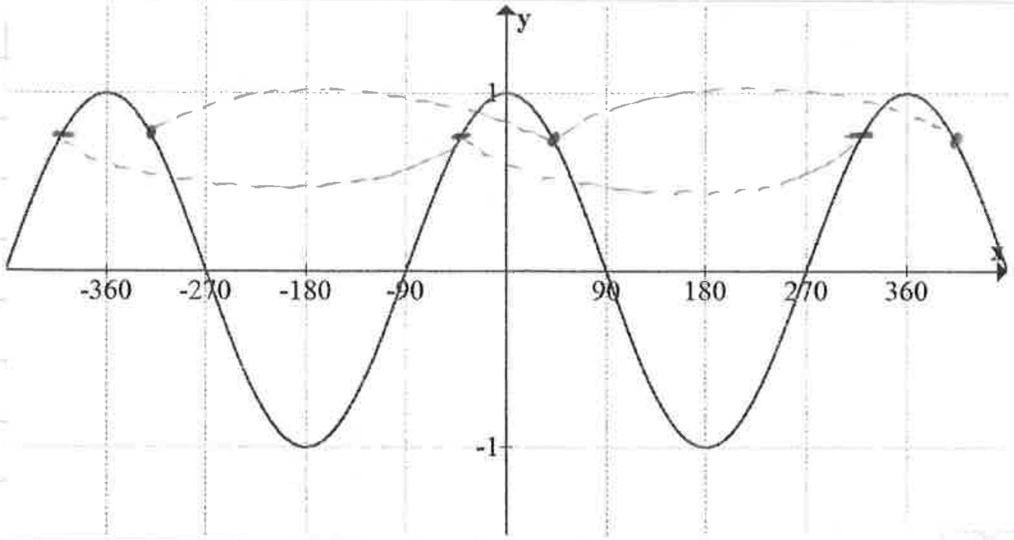
By CAST : $\theta$ is positive, so lies in quadrant I & IV




So $\theta = -41.41^\circ \pm 360^\circ n$ also

for $n = 0, 1, 2, \ldots$

By graph




$\theta = 41.41° \pm 360° n$  (Red dots)

But we see that there is another $\theta$ adjacent to $41.41°$ which also gives $\cos\theta = 3/4$, i.e

$$\theta = -41.41°$$

& This repeats every $360°$. Hence

$$\theta = -41.41° \pm 360° n \quad , n = 0, 1, 2, \ldots$$

(8) $4\cot^2\theta + 12\operatorname{cosec}\theta + 1 = 0$

Using $\cot^2\theta + 1 = \operatorname{cosec}^2\theta$ we have $\cot^2\theta = \operatorname{cosec}^2\theta - 1$
so

$$4(\operatorname{cosec}^2\theta - 1) + 12\operatorname{cosec}\theta + 1 = 0$$

$$\therefore \quad 4\operatorname{cosec}^2\theta - 4 + 12\operatorname{cosec}\theta + 1 = 0$$

hence $\quad 4\operatorname{cosec}^2\theta + 12\operatorname{cosec}\theta - 3 = 0$

By the quadratic formula we have

$$\operatorname{cosec}\theta = \frac{-12 \pm \sqrt{144 + 48}}{8} = -\frac{3}{2} \pm \sqrt{3}$$

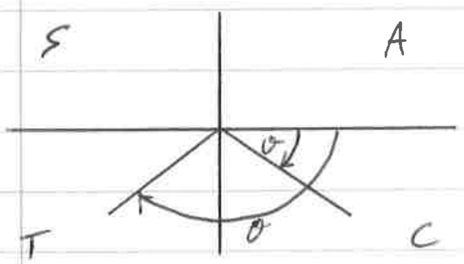
$$= -3.23205081, \; 0.2320508 1$$

So

$$\sin\theta = -0.3094011, \; 4.3094011$$

But $\sin\theta = 4.3094011$ is not valid since $-1 \le \sin\theta \le 1$

By CAST : $\sin\theta = -0.3094011$ is negative so $\theta$ lies in quadrant III and IV




So $\sin\theta = -0.3094011$

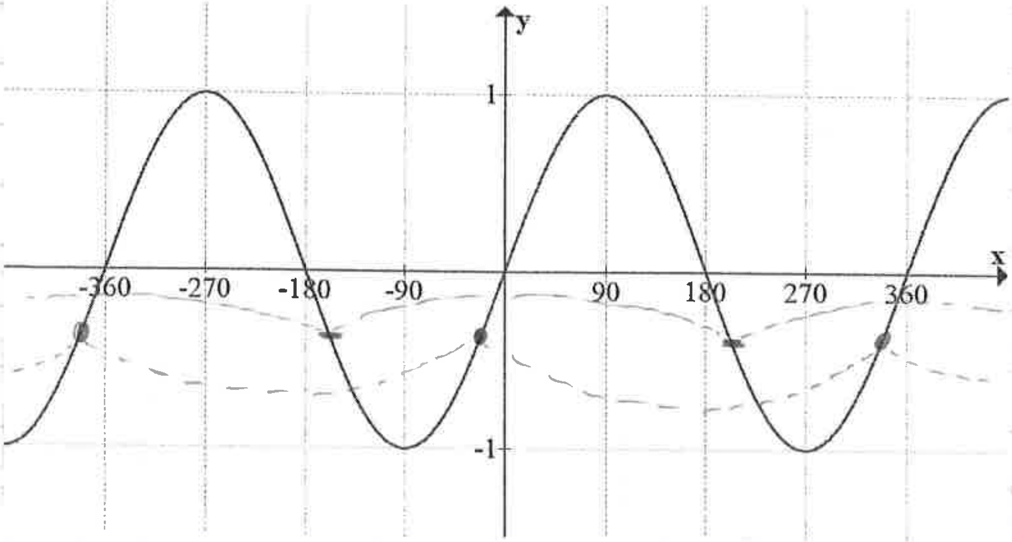
$\theta = -18.02^\circ$

$\Rightarrow$ general sol : $\theta = -18.02^\circ \pm 360^\circ n$

and $\quad \theta = -180^\circ + 18.02^\circ = -161.98^\circ$

$\Rightarrow$ general sol : $\theta = -161.98^\circ \pm 360^\circ n$

By graph




$\sin\theta = -0.3094011$ so $\theta = -18.02° \pm 360° n$

But a 2nd $\theta$ which gives $\sin\theta = -0.3094011$ is
$\theta = -161.98°$ so

$$\theta = -161.98° \pm 360° n$$

(9) $4\sec^2\theta - 3\tan\theta = 5$

Using $1+\tan^2\theta = \sec^2\theta$ we have

$$4(1+\tan^2\theta) - 3\tan\theta = 5$$

$\therefore \quad 4 + 4\tan^2\theta - 3\tan\theta = 5$

$\therefore \quad 4\tan^2\theta - 3\tan\theta - 1 = 0$

$$(4\tan\theta + 1)(\tan\theta - 1) = 0$$

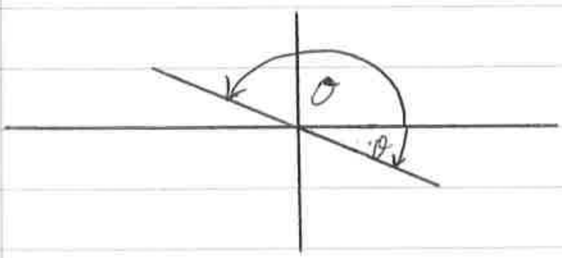
Hence $\quad \tan\theta = -\frac{1}{4},\ 1$

So $\quad \theta = -14.04^\circ \pm 180^\circ n$

and $\quad \theta = +45^\circ \pm 180^\circ n$

By CAST

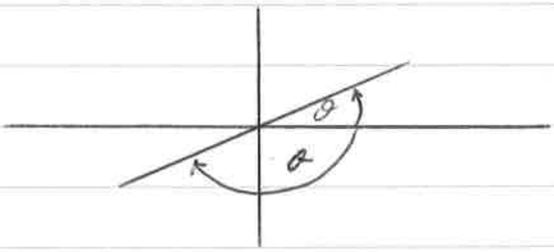
For $\theta = -14.04^\circ$,
$\theta$ is negative so lies in quadrant II & IV




So $\theta = -14.04 + 180^\circ$
$= 165.96^\circ$

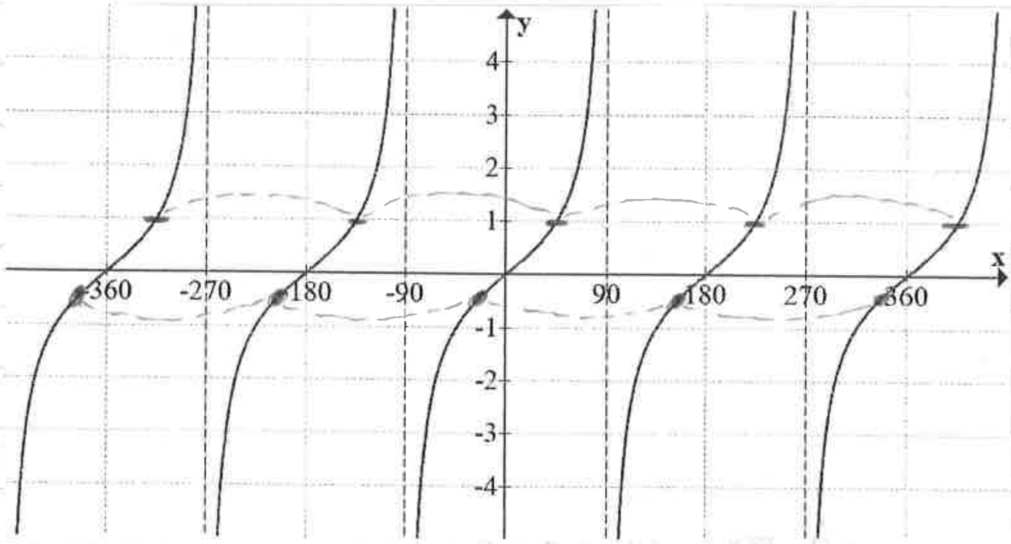
But this is already covered by
$\theta = -14.04^\circ \pm 180^\circ n$
So no change here

For $\theta = 45^\circ$
$\theta$ is positive so lies in quadrant I and III




So $\theta = 45^\circ - 180^\circ = -135^\circ$
But this is already covered by
$\theta = 45^\circ \pm 180^\circ n$
So no change here.

By graph




$\tan\theta = -\frac{1}{4}$ And $\tan\theta = 1$

So $\theta = -14.04^\circ \pm 180^\circ n$ So $\theta = 45^\circ \pm 180^\circ n$

$n = 0, 1, 2, 3, \ldots$

(10) $2\cos\theta - 4\sin^2\theta + 2 = 0$

Using $\sin^2\theta + \cos^2\theta = 1$ we have $\sin^2\theta = 1 - \cos^2\theta$.

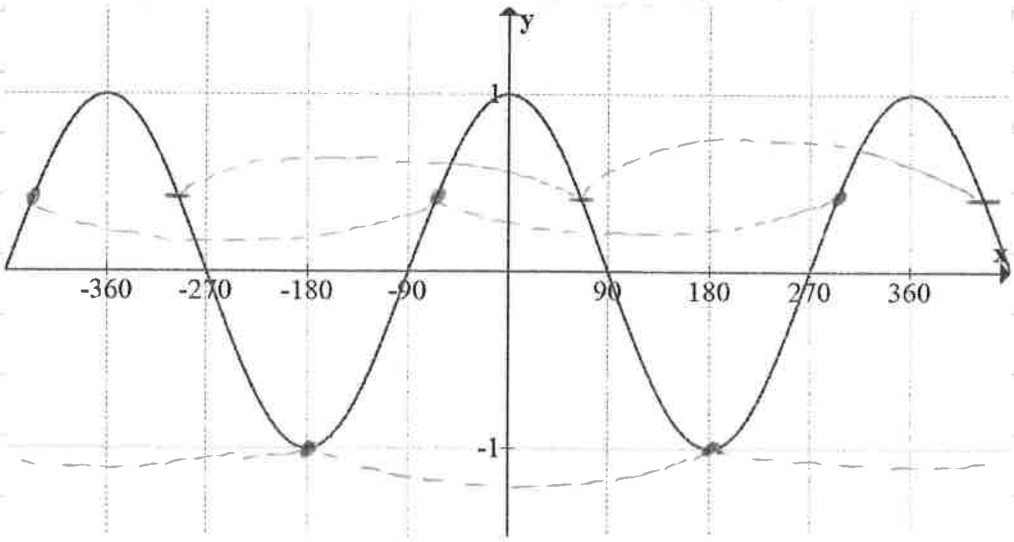
So $\quad 2\cos\theta - 4(1-\cos^2\theta) + 2 = 0$

$$4\cos^2\theta + 2\cos\theta - 2 = 0$$

$$(2\cos\theta - 1)(2\cos\theta + 2) = 0$$

$\therefore \quad \cos\theta = \frac{1}{2}, \ -1$

By graph




$\cos\theta = -1$

So $\quad \theta = 180° \pm 360° n$
$\quad = 180° (1 + 2n)$

$\cos\theta = \frac{1}{2}$

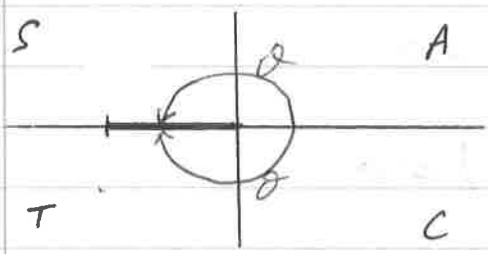
So $\theta = 60°$ and $-60°$ as principal values. So

$$\theta = 60° \pm 360° n$$

and $\theta = -60° \pm 360° n$

By CAST

$\cos\theta = -1$ Means $\theta$ lies in quadrant II & III



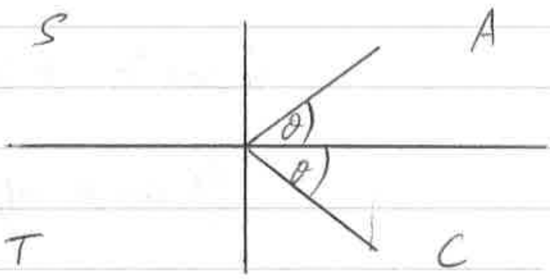

So $\theta = +180^\circ$ and $-180^\circ$

But $-180^\circ$ is covered by

$$\theta = 180^\circ \pm 360^\circ n$$

So Thisis the general Sol.

$\cos\theta = \frac{1}{2}$ means $\theta$ lies in quadrant I and IV




So $\theta = 60^\circ$ and $-60^\circ$

Each Repeats every $360^\circ$ So

$$\theta = \pm 60^\circ \pm 360^\circ n$$

is The general Sol.

(11) $\cot\theta + \tan\theta \equiv \sec\theta \operatorname{cosec}\theta$

$$LHS = \frac{\cos\theta}{\sin\theta} + \frac{\sin\theta}{\cos\theta} = \frac{\cos^2\theta + \sin^2\theta}{\sin\theta\cos\theta}$$

$$RHS = \frac{1}{\cos\theta} \cdot \frac{1}{\sin\theta} = \frac{1}{\sin\theta\cos\theta}$$

So $$\frac{\cos^2\theta + \sin^2\theta}{\sin\theta\cos\theta} = \frac{1}{\sin\theta\cos\theta}$$ ✓

is true since $\cos^2\theta + \sin^2\theta = 1$.

(12) $$\frac{\cos A}{1-\tan A} + \frac{\sin A}{1-\cot A} \equiv \sin A + \cos A$$

LHS:
$$\frac{\cos A}{1-\tan A} = \frac{\cos A}{\frac{\cos A - \sin A}{\cos A}} = \frac{\cos^2 A}{\cos A - \sin A}$$

and
$$\frac{\sin A}{1-\cot A} = \frac{\sin A}{\frac{\sin A - \cos A}{\sin A}} = \frac{\sin^2 A}{\sin A - \cos A}$$

So $$LHS = \frac{\cos^2 A}{\cos A - \sin A} + \frac{\sin^2 A}{\sin A - \cos A}$$

$$= \frac{\cos^2}{\cos A - \sin A} - \frac{\sin^2 A}{\cos A - \sin A}$$

$$= \frac{\cos^2 A - \sin^2 A}{\cos A - \sin A} = \frac{(\cos A - \sin A)(\cos A + \sin A)}{\cos A - \sin A}$$

(by The difference of two squares)

$$\therefore \frac{\cos A}{1-\tan A} + \frac{\sin A}{1-\cot A} = \sin A + \cos A. \checkmark$$

(13) $\tan^2\theta + \cot^2\theta \equiv \sec^2\theta + \operatorname{cosec}^2\theta - 2$

$$\therefore \tan^2\theta - \sec^2\theta = \operatorname{cosec}^2\theta - \cot^2\theta - 2$$

By $1+\tan^2\theta = \sec^2\theta$ and $\cot^2\theta + 1 = \operatorname{cosec}^2\theta$
we have

$$\tan^2\theta - (1+\tan^2\theta) = \operatorname{cosec}^2\theta - (\operatorname{cosec}^2\theta - 1) - 2$$

$$-1 = -1 \checkmark \text{ is true } \forall \theta.$$

(14) $\dfrac{\sin A}{1+\cos A} \equiv \dfrac{1-\cos A}{\sin A}$

Using the hint: $\dfrac{\sin A}{1+\cos A} \cdot \dfrac{1-\cos A}{1-\cos A} \equiv \dfrac{(1-\cos A)\cdot \sin A}{1-\cos^2 A}$

using $\cos^2 A + \sin^2 A = 1$ we have $\sin^2 A = 1-\cos^2 A$
So

$$\frac{\sin A}{1+\cos A} \cdot \frac{1-\cos A}{1-\cos A} \equiv \frac{(1-\cos A)\sin A}{\sin^2 A}$$

$$\equiv \frac{1-\cos A}{\sin A} \equiv RHS \checkmark$$

Another way: cross-multiply. So

$$\frac{\sin A}{1+\cos A} \equiv \frac{1-\cos A}{\sin A} \quad \text{becomes} \quad \sin^2 A \equiv (1-\cos A)(1+\cos A)$$

$$\equiv 1-\cos^2 A$$

$$\equiv \sin^2 A \quad \checkmark$$

(15) $(\sec^2\theta + \tan^2\theta)(\operatorname{cosec}^2\theta + \cot^2\theta) \equiv 1 + 2\sec^2\theta \operatorname{cosec}^2\theta$

LHS: by $\left.\begin{array}{l} 1+\tan^2\theta = \sec^2\theta \\ \text{and } \cot^2\theta + 1 = \operatorname{cosec}^2\theta \end{array}\right\} \circledast$

we have $[(1+\tan^2\theta)+\tan^2\theta][(\cot^2\theta+1)+\cot^2\theta]$

$$= (1+2\tan^2\theta)(1+2\cot^2\theta)$$

$$= 1 + 2\cot^2\theta + 2\tan^2\theta + 4\tan^2\theta\cot^2\theta$$

Seems we are going in the wrong direction with our use of $\circledast$ so instead use $\circledast$ as

$$\tan^2\theta = \sec^2\theta - 1 \quad \& \quad \cot^2\theta = \operatorname{cosec}^2\theta - 1$$

Hence we have, for LHS

$$[\sec^2\theta + (\sec^2\theta - 1)][\operatorname{cosec}^2\theta + (\operatorname{cosec}^2\theta - 1)]$$

$$= (2\sec^2\theta - 1)(2\operatorname{cosec}^2\theta - 1)$$

$$= 4\sec^2\theta\operatorname{cosec}^2\theta - 2\sec^2\theta - 2\operatorname{cosec}^2\theta + 1$$

$$= 4\sec^2\theta\operatorname{cosec}^2\theta - 2(\sec^2\theta + \operatorname{cosec}^2\theta) + 1$$

$$= 4\sec^2\theta\,\mathrm{cosec}^2\theta - 2\left(\frac{1}{\cos^2\theta} + \frac{1}{\sin^2\theta}\right) + 1$$

$$= 4\sec^2\theta\,\mathrm{cosec}^2\theta - 2\left(\frac{\cos^2\theta + \sin^2\theta}{\cos^2\theta\sin^2\theta}\right) + 1$$

$$= 4\sec^2\theta\,\mathrm{cosec}^2\theta - 2\left(\frac{1}{\cos^2\theta\sin^2\theta}\right) + 1$$

$$= 4\sec^2\theta\,\mathrm{cosec}^2\theta - 2\sec^2\theta\,\mathrm{cosec}^2\theta + 1$$

$$= 2\sec^2\theta\,\mathrm{cosec}^2\theta + 1 \quad \checkmark$$

(16) $$\frac{\sin A}{1+\cos A} + \frac{1+\cos A}{\sin A} \equiv \frac{2}{\sin A}$$

$$\sin^2 A + (1+\cos A)^2 \equiv \frac{2\sin A\,(1+\cos A)}{\sin A}$$

$$\sin^2 A + 1 + 2\cos A + \cos^2 A \equiv 2(1+\cos A)$$

$$1 + 1 + 2\cos A = 2 + 2\cos A \equiv 2(1+\cos A) \checkmark$$

(17) $$\sec^2 A \equiv \frac{\mathrm{cosec}\,A}{\mathrm{cosec}\,A - \sin A}$$

By $1+\tan^2 A = \sec^2 A$ we have

$$1+\tan^2 A \equiv \frac{\mathrm{cosec}\,A}{\mathrm{cosec}\,A - \sin A}$$

So $$\frac{\cos^2 A + \sin^2 A}{\cos^2 A} \equiv \frac{\mathrm{cosec}\,A}{\mathrm{cosec}\,A - \sin A} \quad \left(\text{via } \tan A = \frac{\sin A}{\cos A}\right)$$

So $$\frac{1}{\cos^2 A} \equiv \frac{\operatorname{cosec} A}{\operatorname{cosec} A - \sin A}$$

via $\cos^2 A + \sin^2 A = 1$.

From RHS we have $$\frac{\operatorname{cosec} A}{\operatorname{cosec} A + \sin A} = \frac{1/\sin A}{\frac{1}{\sin A} - \sin A}$$

$$= \frac{1/\sin A}{\frac{1 - \sin^2 A}{\sin A}}$$

$$= \frac{1}{1 - \sin^2 A} = \frac{1}{\cos^2 A}$$

Hence $$\frac{1}{\cos^2 A} \equiv \frac{1}{\cos^2 A} \checkmark \quad (\text{i.e. } LHS \equiv RHS)$$

(18) $$(1 + \sin\theta + \cos\theta)^2 \equiv 2(1 + \sin\theta)(1 + \cos\theta)$$

LHS: $$1 + 2(\sin\theta + \cos\theta) + (\sin\theta + \cos\theta)^2$$

$$= 1 + 2\sin\theta + 2\cos\theta + \sin^2\theta + 2\sin\theta\cos\theta + \cos^2\theta$$

By $\cos^2\theta + \sin^2\theta = 1$ we have

$$1 + 2(\sin\theta + \cos\theta) + (\sin\theta + \cos\theta)^2 = 2 + 2\sin\theta + 2\cos\theta + 2\sin\theta\cos\theta$$

$$= 2(1 + \sin\theta + \cos\theta + \sin\theta\cos\theta)$$

$$= 2(1 + \sin\theta)(1 + \cos\theta) \equiv RHS \checkmark$$

(19) $\dfrac{\tan^2 A + \cos^2 A}{\sin A + \sec A} \equiv \sec A - \sin A$

It is tempting to work the LHS & try to simplify it to the RHS. If you try this you end up with hours spent on algebra & involving terms using $\cos^4\theta$. This is much too much effort.

So we try to find a way to work the RHS. It would be nice if we had a $\sec^2 A$ & a $\sin^2 A$. The only way to get these is to multiply RHS by either $(\sec A + \sin A)$ or $(\sec A - \sin A)$ in some specific form.

We will use the former method. (Why?)

So $\sec A - \sin A = \sec A - \sin A \cdot \dfrac{\sec A + \sin A}{\sec A + \sin A}$

$$= \frac{\sec^2 A - \sin^2 A}{\sec A + \sin A}.$$

By $1 + \tan^2 A = \sec^2 A$ and $\cos^2 A + \sin^2 A = 1$ we have

$$\frac{\sec^2 A - \sin^2 A}{\sec A + \sin A} = \frac{1 + \tan^2 A - (1 - \cos^2 A)}{\sec A + \sin A}$$

$$= \frac{\tan^2 A + \cos^2 A}{\sec A + \sin A} \equiv \text{LHS} \checkmark$$

(20) (a) $x = 4\sec\theta \longrightarrow x/4 = \sec\theta$

$y = 5\tan\theta \longrightarrow y/5 = \tan\theta$

So $\left(\frac{x}{4}\right)^2 = \sec^2\theta$ & $\left(\frac{y}{5}\right)^2 = \tan^2\theta$

$\left(\frac{x}{4}\right)^2 = 1+\tan^2\theta$ & $\left(\frac{y}{5}\right)^2 = \tan^2\theta$

$\left(\frac{x}{4}\right)^2 - 1 = \tan^2\theta$ & $\left(\frac{y}{5}\right)^2 = \tan^2\theta$

So $\frac{x^2}{16} - 1 = \frac{y^2}{25} \Rightarrow \frac{x^2}{16} - \frac{y^2}{25} = 1$

(Q: Is This a circle, Ellipse or hyperbola? What is its "centre"?)

(b) $x = a\,\mathrm{cosec}\,\theta \longrightarrow x/a = \mathrm{cosec}\,\theta$

$y = b\cot\theta \longrightarrow y/b = \cot\theta$

So $\left(\frac{x}{a}\right)^2 = \mathrm{cosec}^2\theta$ & $\left(\frac{y}{b}\right)^2 = \cot^2\theta$

$\left(\frac{x}{a}\right)^2 = \cot^2\theta + 1$ & $\left(\frac{y}{b}\right)^2 = \cot^2\theta$

$\therefore \left(\frac{x}{a}\right)^2 - 1 = \cot^2\theta$ & $\left(\frac{y}{b}\right)^2 = \cot^2\theta$

hence

$\frac{x^2}{a^2} - 1 = \frac{y^2}{b^2} \Rightarrow \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

(c) $x = 2\tan\theta \longrightarrow x/2 = \tan\theta$

$y = 3\cos\theta \longrightarrow y/3 = \cos\theta$

So $\left(\frac{x}{2}\right)^2 = \tan^2\theta$ & $\left(\frac{y}{3}\right)^2 = \cos^2\theta$

$\left(\frac{x}{2}\right)^2 = \sec^2\theta - 1$ & $\left(\frac{y}{3}\right)^2 = \cos^2\theta$

$\therefore \left(\frac{x}{2}\right)^2 + 1 = \sec^2\theta$ & $\left(\frac{3}{y}\right)^2 = \sec^2\theta$

hence

$$\frac{x^2}{4} + 1 = \frac{9}{y^2} \Rightarrow \frac{x^2}{4} - \frac{9}{y^2} = -1$$

$$\Rightarrow x^2y^2 - 36 = -4y^2$$

$$\therefore x^2y^2 + 4y^2 = 36$$

$$y^2(x^2+4) = 36$$

(d) $x = 1 - \sin\theta \longrightarrow x - 1 = -\sin\theta \longrightarrow 1 - x = \sin\theta$

$y = 1 + \cos\theta \longrightarrow y - 1 = \cos\theta$

So $(1-x)^2 = \sin^2\theta$ & $(y-1)^2 = \cos^2\theta$

add: $(1-x)^2 + (y-1)^2 = \sin^2\theta + \cos^2\theta$

$\Rightarrow (1-x)^2 + (y-1)^2 = 1$

(Q: what geometric shape is this? what is its centre?)

(e) $x = 2 + \tan\theta \longrightarrow x - 2 = \tan\theta$

$y = 2\cos\theta \longrightarrow \frac{y}{2} = \cos\theta$

So $(x-2)^2 = \tan^2\theta$ & $\left(\frac{y}{2}\right)^2 = \cos^2\theta$

$(x-2)^2 = \sec^2\theta - 1$ & $\left(\frac{y}{2}\right)^2 = \cos^2\theta$

$(x-2)^2 + 1 = \sec^2\theta$ & $\left(\frac{y}{2}\right)^2 = \cos^2\theta$

$\left(\frac{2}{y}\right)^2 = \sec^2\theta$

Hence

$$(x-2)^2 + 1 = \left(\frac{2}{y}\right)^2$$

(f) $x = a\sec\theta \longrightarrow x/a = \sec\theta$

$y = b\sin\theta \longrightarrow y/b = \sin\theta$

So $\left(\frac{x}{a}\right)^2 = \sec^2\theta$ & $\left(\frac{y}{b}\right)^2 = \sin^2\theta$

$\left(\frac{a}{x}\right)^2 = \cos^2\theta$ & $\left(\frac{y}{b}\right)^2 = \sin^2\theta$

add : $\left(\frac{a}{x}\right)^2 + \left(\frac{y}{b}\right)^2 = \cos^2\theta + \sin^2\theta$

So $\left(\frac{a}{x}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$

(21)

(a) $$\frac{1-\sec^2 A}{1-\operatorname{cosec}^2 A} = \frac{1-(1+\tan^2 A)}{1-(\cot^2 A+1)}$$

$$= \frac{\tan^2 A}{\cot^2 A} = \tan^4 A$$

(b) $$\frac{\sin\theta}{\sqrt{1-\cos^2\theta}} = \frac{\sin\theta}{\sqrt{\sin^2\theta}} = 1$$

(c) $$\frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta} = \frac{\sin^2\theta+\cos^2\theta}{\sin\theta\cos\theta}$$

$$= \frac{1}{\sin\theta\cos\theta} = \sec\theta\operatorname{cosec}\theta$$

(d) $$\frac{\sqrt{1+\tan^2\theta}}{\sqrt{1-\sin^2\theta}} = \frac{\sqrt{\sec^2\theta}}{\sqrt{\cos^2\theta}}$$

From $1+\tan^2\theta = \sec^2\theta$

From $\cos^2\theta + \sin^2\theta = 1$

$$= \frac{\sec\theta}{\cos\theta} = \sec^2\theta$$

(e) $$\frac{1}{\cos\theta\sqrt{1+\cot^2\theta}} = \frac{1}{\cos\theta\sqrt{\operatorname{cosec}^2\theta}}$$

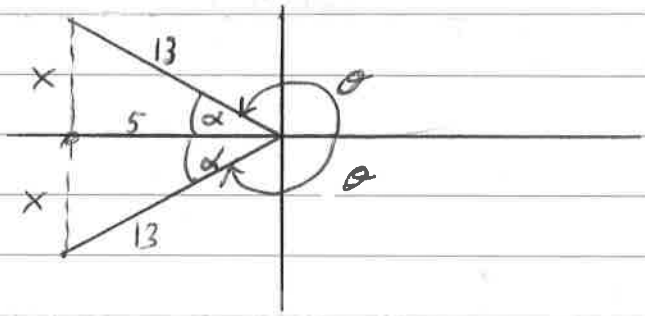
From $1+\cot^2\theta = \operatorname{cosec}^2\theta$

$$= \frac{1}{\cos\theta\operatorname{cosec}\theta}$$

$$= \frac{\sin\theta}{\cos\theta} = \tan\theta.$$

(22) (a) $\cos\theta = -\frac{5}{13}$

Cos is Negative, so $\theta$ lies in quadrant II & III




From diagram we have $13^2 = x^2 + 12^2 \Rightarrow x = \pm 12$

But $\theta$ is Reflex (see question), measured clockwise from $0°$ (i.e. in the $-\theta$ direction), so $x = -12$

$\therefore \sin\theta = -\frac{12}{13}$

Since $\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{-12/13}{-5/13}$,

$\tan\theta = \frac{12}{5}$

OR $\cos^2\alpha + \sin^2\alpha = 1 \Rightarrow \sin^2\alpha = \sqrt{1-\cos^2\alpha}$

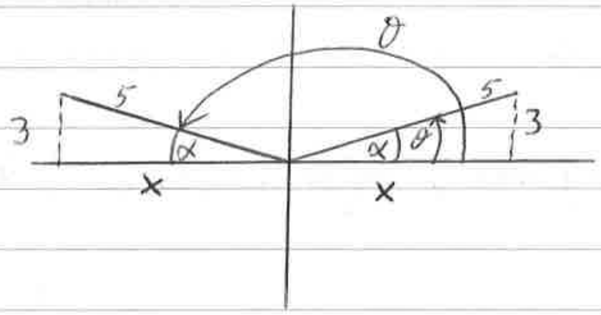
So $\sin^2\alpha = \pm\sqrt{1 - \frac{25}{169}} = \pm\frac{12}{13}$

But $\theta$ is Reflex so: $\sin\theta = -\frac{12}{13}$

and $\therefore \tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{12}{5}$

(b) $\sin\theta = \frac{3}{5}$

Sin is positive, so $\theta$ lies in quadrant I & II.




From the diagram we have $5^2 = 3^2 + x^2 \Rightarrow x = \pm 4$

But $\theta$ is obtuse $\Rightarrow x = -4$

So $\cos\theta = -\frac{4}{5}$.

Since $\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{3/5}{-4/5}$,

$\tan\theta = -\frac{3}{4}$.

OR $\cos^2\alpha + \sin^2\alpha = 1 \Rightarrow \cos\alpha = \pm\sqrt{1 - \sin^2\alpha}$

So $\cos\alpha = \pm\sqrt{1 - \frac{9}{25}} = \pm\frac{4}{5}$

But $\theta$ is obtuse $\cos\theta = -\frac{4}{5}$

& $\tan\theta = \frac{\sin\theta}{\cos\theta} = -\frac{3}{4}$

(c) $\tan\theta = \frac{7}{24}$

tan is positive, so $\theta$ lies in quadrant I & III
But $\theta$ is acute, so lies in quadrant I

$\therefore$ By pythagoras, hypothenuse $= \pm\sqrt{7^2+24^2}$

$= \pm 25$

But $\theta$ lies in quadrant I
So

hypothenuse $= +25$

$\therefore \sin\theta = \frac{7}{25}$ & $\cos\theta = \frac{24}{25}$

or $1+\tan^2\theta = \sec^2\theta$

So $\sec\theta = \pm\sqrt{1+\tan^2\theta} = \pm\sqrt{1+\frac{7^2}{24^2}}$

$= \pm\frac{25}{24}$

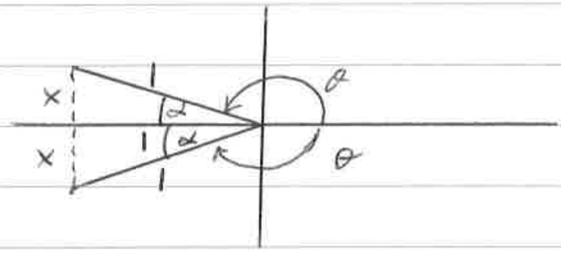
So $\cos\theta = \pm\frac{24}{25}$

But $\theta$ is acute so $\cos\theta$ is positive : $\cos\theta = \frac{24}{25}$

& $\sin\theta = \tan\theta \cdot \cos\theta = \frac{7}{24} \cdot \frac{24}{25} = \frac{7}{25}$.

(d) $\cos\theta = -1$

cos is negative $\Rightarrow \theta$ lies in quadrant II & III




From diagram $1^2 = 1^2 + x^2 \Rightarrow x = 0$

$\therefore$ hypothenuse & adjacent lie on same straight line

$\cos\theta = \frac{-1}{1}$ So $\sin\theta = \frac{0}{1} = 0$

and $\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{0}{1} = 0.$